Solutions

Practice Exam 3
Sections 3.3-3.4 and 4.1-4.3

Treat this like an exam. Answer the following questions. *You must show your work to receive full credit.*

1a. . Jim buys a car. Every year, Jims car depreciates (loses) 12% of its value from the previous year. When the car is new (year 0), it is valued at $26,000. Give a recurrence relation for the value of Jims car after n years for n0. You must use correct notation.

$$P(t) = \begin{cases} 26{,}000, & t=0 \\ (.88)\cdot P(t-1), & t>0 \end{cases}$$

1b. How many ways are there to rearrange the word SCARF?

All letters are different.

$P(5,5) = 5!$ ways.

1c. How many ways are there to rearrange the word MISSISSIPPI?

1 M, 4 S's, 4 I's, 2 P's = 11 letters

(1) Place the 1 M: $C(11,1) = 11$ ways
(2) Place the 4 S's: $C(10,4) = 210$
(3) Place the 4 I's: $C(6,4) = 15$ ways
(4) Place the 2 P's: $C(2,2) = 1$ way

$\Rightarrow 11 \cdot 210 \cdot 15 \cdot 1$

$= 34{,}650$ ways

2. A committee is made up of 7 women and 4 men.

(a) How many ways are there to form a subcommittee of 3 people?

(b) How many ways are there to form a subcommittee of 3 people if the subcommittee must contain at least 2 women?

(c) How many different linear seating arrangements of the committee are there?

(d) How many different linear seating arrangements of the committee are there if the 4 men must sit together?

(e) How many different linear seating arrangements of the committee are there if none of the 4 men are allowed to sit together?

(a) 11 people. $C(11,3)=165$ ways

(b) Case 1:
(1) Choose two women: $C(7,2)=21$ ways
(2) Choose one man: $C(4,1)=4$ ways
$\Rightarrow 21\cdot 4 = 84$ ways

Case 2: Choose three women: $C(7,3)=35$

In total, $84+35=119$ ways.

(c) Total Seating Arrangements: $P(11,11)=11!$

(d) (1) Group the 4 men together. There are $P(4,4)=4!$ ways to do this
(2) Order the women. There are $P(7,7)=7!$ ways to do this.
(3) Place the group of men amongst the women. There are $C(8,1)=8$ ways.
Multiplication Principle gives $4!\cdot 7!\cdot 8$ seating arrangements

(e) (1) Order the men and women. $4!\cdot 7!$ ways.
~~The first man must sit~~
(2) Place the men in the 8 spaces between the women (this includes outside spaces.)
There are $C(8,4)=70$ ways.
So, in total, $4!\cdot 7!\cdot 70$ seating arrangements

3. The following problem refers to strings from the English alphabet $\{A, B, , Z\}$.

(a) How many 4-letter strings are possible?

(b) How many 4-letter strings are possible if no letter can be repeated?

(c) How many 4-letter strings are there that end with the letter B?

(a) 26^4

(b) $P(26,4) = \frac{26!}{22!}$

(c) B is fixed, so we need only choose the first 3 letters.

26^3

4. Prove Theorem 4.1 from the text. (This is Theorem 1 from Worksheet Section 4.1.)

We will show by induction that for any collection ~~Let~~ $A_1, A_2, \ldots, A_n$ ~~be~~ of pairwise disjont (pwd) finite sets, that

~~We~~ $$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Base Case: For any pwd finite sets A_1 and A_2,

$$|A_1 \cup A_2| = |A_1| + |A_2| \text{ by the Addition Principle.}$$

Inductive Hypothesis: Suppose every collection of $K-1$ pwd finite sets satisfies $|A_1 \cup A_2 \cup \cdots \cup A_{K-1}| = |A_1| + |A_2| + \cdots + |A_{K-1}|$ for some $K > 2$.

Inductive Step: Let $A_1, A_2, \ldots, A_K$ be a collection of pwd finite sets. Then, letting $A = A_1 \cup A_2 \cup \cdots \cup A_{K-1}$, we have

$$|A \cup A_K| = |A| + |A_K| \quad \text{(by the base case)}$$

$$= (|A_1| + |A_2| + \cdots + |A_{K-1}|) + |A_K| \quad \text{(by the inductive hypothesis}$$

5. Prove Theorem 4.2 from the text. (This is Theorem 2 from Worksheet Section 4.1.)

Repeat exactly from the previous problem with

(i) $\cup$ replaced by $\times$,

(ii) $+$ replaced by $\cdot$, and

(iii) Addition Principle replaced by Multiplication Principle.

6. In the following suppose that a *license plate* is made up of 6 entries from the alphabet $\{A, B, \ldots, Z\} \cup \{0, 1, \ldots, 9\}$.

(a) How many license plates are there which have 3 letters followed by 3 numbers?

(b) How many license plates are there which contain exactly 4 letters?

(c) How many palindromic license plates are there?

(d) How many palindromic license plates are there with 4 numbers?

(a) $26^3 \cdot 10^3$ license plates

(b) (1) Choose the letters: 26^4
(2) Place the letters: $C(6,4) = 15$
(3) Choose the numbers: 10^2
(4) Place the numbers: $C(2,2) = 1$

$\Rightarrow 26^4 \cdot 15 \cdot 10^2$ license plates

(c) We only need to choose the first 3 entries and the last 3 are fixed. 36^3 license plates.

(d) Same as above, only need to work with half the entries.

(1) Choose the letters: 26^2
(2) Place the letters: $C(3,2) = 3$
(3) Choose the number: 10^1
(4) Place the number: $C(1,1) = 1$

$\Rightarrow 26^2 \cdot 3 \cdot 10$ License plates

7a. Suppose that $f : X \to Y$ is a seven-to-one function mapping X onto Y and that $|Y| = 6$. What is $|X|$? Give a brief explanation of your answer. If there is not enough information, state so and explain why.

~~By~~ f is seven-to-one and onto so

$$|X| = 7|Y| = 7 \cdot 6 = 42$$

7b. Suppose that $f : X \to Y$ is a one-to-one correspondence and that $|X| = 8$. What is $|Y|$? Give a brief explanation of your answer. If there is not enough information, state so and explain why.

f is a one-to-one correspondeace so

$$|Y| = |X| = 8.$$

7c. Suppose that $f : X \to Y$ is a two-to-one function and that $|X| = \overset{8}{\cancel{7}}$. What is $|Y|$? Give a brief explanation of your answer. If there is not enough information, state so and explain why.

If f were two-to-one <u>and</u> onto then

$8 = |X| = 2|Y|$ and so would equal $|Y| = 4$.

However, we do not know that f is onto so the best we can do is

$$|Y| \geq 4.$$

8a. Explain why, in a class of 32, there will always be a group of at least 5 students who were born on the same day of the week.

Since $\lceil \frac{32}{7} \rceil = 5$, the generalized Pigeonhole Principle tells us so.

8b. Explain why for every 27-word sequence in the US Constitution, at least two words will start with the same letter.

Since $27 > 26$, the Pigeonhole Principle tells us so.